ON A CERTAIN CORRESPONDENCE BETWEEN SURFACES IN HYPERSPACE*

BY

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1. Introduction

Consider a surface S and a point x on S. Let the parametric vector equation of S be

$$x = x(u, v). \tag{1}$$

The ambient space of the osculating planes at the point x to all of the curves through x is a certain space $S(2, 0)$ called *the two-osculating space of S at x.* This space is determined by the six points

$$x,\ x_u,\ x_v,\ x_{uu},\ x_{uv},\ x_{vv}. \tag{2}$$

It is the purpose of this paper to find all surfaces $\bar{S}$ in one-to-one point correspondence with S, such that the two-osculating space $\bar{S}(2, 0)$ of $\bar{S}$ coincides with the two-osculating space $S(2, 0)$ of S at corresponding points. We shall find that the surface S is not arbitrary, but that the functions x satisfy certain third-order partial differential equations studied by Lane† and by Bompiani.‡ A similar statement holds for the surface $\bar{S}$.

Let the surfaces S and $\bar{S}$ be in one-to-one point correspondence so that the corresponding points have the same curvilinear coordinates.

In order that $\bar{S}(2, 0)$ at $\bar{x}$ coincide with $S(2, 0)$ at x, it is necessary and sufficient that the functions

$$\bar{x},\ \bar{x}_u,\ \bar{x}_v,\ \bar{x}_{uu},\ \bar{x}_{uv},\ \bar{x}_{vv} \tag{3}$$

be expressible as linear, homogeneous functions of the functions (2). The parametric vector equation of $\bar{S}$ will therefore be of the form

$$\bar{x} = \bar{x}(u, v) = Ax_{uu} + Bx_{uv} + Cx_{vv} + \alpha x_u + \beta x_v + \gamma x. \tag{4}$$

We shall call the case in which $S(2, 0)$ is a space of five dimensions and in which the coefficients A, B, C of (4) satisfy the inequality

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† E. P. Lane, *Integral surfaces of pairs of partial differential equations of the third order*, these Transactions, vol. 32 (1930), pp. 782–793. Hereafter referred to as Lane, *Surfaces*.

‡ E. Bompiani, *Determinazione delle superficie integrali d'un sistema di equazioni a derivate parziali lineari ed omogenee*, Rendiconti del Reale Istituto Lombardo di Scienze e Lettere, vol. 52 (1919), pp. 820–830. Hereafter referred to as Bompiani, *Surfaces*.

(5) $$B^2 - 4AC \neq 0$$

the non-parabolic case, and the case in which $S(2, 0)$ is a space of five dimensions and in which

(6) $$B^2 - 4AC = 0$$

the parabolic case. By proper choice of ϕ, ψ, and λ in the transformation

$$\bar{u} = \phi(u, v), \quad \bar{v} = \psi(u, v), \quad \bar{x} = \lambda \bar{x}',$$

in the non-parabolic case, we may write (4) in the form

(7) $$\bar{x} = x_{uv} + \alpha x_u + \beta x_v + \gamma x;$$

and in the parabolic case in the form

(8) $$\bar{x} = x_{uu} + \alpha x_u + \beta x_v + \gamma x.$$

We shall denote by $S(3, 0)$ the ambient space of the three-dimensional spaces osculating all of the curves on S through x. The space $S(3, 0)$ is determined by the six points (2) and the points

(9) $$x_{uuu},\ x_{uuv},\ x_{uvv},\ x_{vvv}.$$

2. The non-parabolic case

If we differentiate $\bar{x}$ defined by (7) with respect to u and v we obtain the following expressions:

(10) $$\begin{aligned} \bar{x}_u &= x_{uuv} + \alpha x_{uu} + \beta x_{uv} + (\alpha_u + \gamma) x_u + \beta_u x_v + \gamma_u x, \\ \bar{x}_v &= x_{uvv} + \alpha x_{uv} + \beta x_{vv} + \alpha_v x_u + (\beta_v + \gamma) x_v + \gamma_v x. \end{aligned}$$

The points $\bar{x}_u$, $\bar{x}_v$ are in the space $S(2, 0)$ if, and only if, the functions x defining the surface S satisfy a system of differential equations of the form

(11) $$\begin{aligned} x_{uuv} &= a x_{uu} + h x_{uv} + b x_{vv} + l x_u + m x_v + d x, \\ x_{uvv} &= a' x_{uu} + h' x_{uv} + b' x_{vv} + l' x_u + m' x_v + d' x. \end{aligned}$$

It follows therefore *that in the non-parabolic case* $S(3, 0)$ *is of dimensions no higher than seven.*

Subcase a. Suppose that $S(3, 0)$ is a space of seven dimensions. It follows that the functions x satisfy the equations (11) and no other third-order differential equations. Under these conditions some of the integrability conditions* of system (11) are

$$a' = b = 0,\ ah' + l' - a^2 - a_v = 0,\ b'h + m - b'^2 - b'_u = 0.$$

Equations (10) may be written in the form

* Bompiani, *Surfaces*, p. 632.

$$(12)\quad \begin{aligned} \bar{x}_u &= (a+\alpha)x_{uu} + (h+\beta)x_{uv} + (l+\alpha_u+\gamma)x_u + (m+\beta_u)x_v + (d+\gamma_u)x,\\ \bar{x}_v &= (h'+\alpha)x_{uv} + (b'+\beta)x_{vv} + (l'+\alpha_v)x_u + (m'+\beta_v+\gamma)x_v + (d'+\gamma_v)x. \end{aligned}$$

From (12) we see that the points $\bar{x}_{uu}$, $\bar{x}_{uv}$, $\bar{x}_{vv}$ lie in $S(2, 0)$ if, and only if,

$$(13)\qquad \alpha + a = 0, \quad \beta + b' = 0.$$

Therefore *the point* $\bar{x}$ *defined by the expression*

$$(14)\qquad \bar{x} = x_{uv} - ax_u - b'x_v + \gamma x$$

generates a surface $\bar{S}$ *whose two-osculating space* $\bar{S}(2, 0)$ *at* $\bar{x}$ *coincides with the two-osculating space* $S(2, 0)$ *at* x.

From (12) and (14) we find that the expressions for $\bar{x}_u$ and $\bar{x}_v$ may be written in the form

$$(15)\quad \begin{aligned} \bar{x}_u &= [a(h-b') + l - a_u + \gamma]x_u + [d + \gamma_u + \gamma(h-b')]x + (h-b')\bar{x},\\ \bar{x}_v &= [b'(h'-a) + m' - b'_v + \gamma]x_v + [d' + \gamma_v + \gamma(h'-a)]x + (h'-a)\bar{x}. \end{aligned}$$

Therefore *the lines g joining corresponding points* x *and* $\bar{x}$ *of* S *and* $\bar{S}$ *form a congruence G, and the surfaces* S *and* $\bar{S}$ *sustain C nets* in relation C; the developables of G intersect* S *and* $\bar{S}$ *in these C nets. Conversely if two nets are in relation C their sustaining surfaces have coincident two-osculating spaces at corresponding points.*

Subcase b. Suppose that $S(3, 0)$ is of six dimensions. By proper choice of the notation, the functions x satisfy a system of differential equations of the form

$$(16)\quad \begin{aligned} x_{uuv} &= ax_{uu} + hx_{uv} + bx_{vv} + lx_u + mx_v + dx,\\ x_{uvv} &= a'x_{uu} + h'x_{uv} + b'x_{vv} + l'x_u + m'x_v + d'x\\ x_{uuu} &= Ax_{vvv} + a''x_{uu} + h''x_{uv} + b''x_{vv} + l''x_u + m''x_v + d''x, \end{aligned}$$

but no other third-order differential equations.

From (7) we find that

$$(17)\quad \begin{aligned} \bar{x}_u &= (a+\alpha)x_{uu} + (h+\beta)x_{uv} + bx_{vv} + (l+\alpha_u+\gamma)x_u + (m+\beta_u)x_v\\ &\quad + (d+\gamma_u)x,\\ \bar{x}_v &= a'x_{uu} + (h'+\alpha)x_{uv} + (\beta+b')x_{vv} + (l'+\alpha_v)x_u + (m'+\beta_v+\gamma)x_v\\ &\quad + (d'+\gamma_v)x. \end{aligned}$$

It follows from (17) and (16) that the points $\bar{x}_{uu}$, $\bar{x}_{uv}$, $\bar{x}_{vv}$ lie in $S(2, 0)$ if, and only if,

$$(18)\qquad \beta + b' = 0, \quad b = 0, \quad A(a+\alpha) = 0.$$

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* V. G. Grove, *The transformation C of nets in hyperspace*, these Transactions, vol. 33 (1931), pp. 733–741.

If we use (18) we may write equations (17) in the form

$$
(19)\quad \begin{aligned}
\bar{x}_u &= (a+\alpha)x_{uu} + [(l+\alpha_u+\gamma) - \alpha(h-b')]x_u + [m - b'_u + b'(h-b')]x_v \\
&\quad + [d+\gamma_u - \gamma(h-b')]x + (h-b')\bar{x}, \\
\bar{x}_v &= a'x_{uu} + [l' + \alpha_v - \alpha(h'+\alpha)]x_u + [m' - b'_v + \gamma + b'(h'+\alpha)]x_v \\
&\quad + [d' + \gamma_v - \gamma(h'+\alpha)]x + (h'+\alpha)\bar{x}.
\end{aligned}
$$

Some of the integrability conditions of system (16) with $b=0$ are

$$
(20)\quad \begin{gathered}
Aa' = 0,\ a^2 + a'h + a_v = a'a'' + ah' + a'b' + a'_u + l', \\
b'(h-b') + m - b'_u = a'b''.
\end{gathered}
$$

A. Suppose first that $A\neq 0$, $a'=0$. Under conditions (18) equations (19) may be written in the form

$$
(21)\quad \begin{aligned}
\bar{x}_u &= [l - a_u + \gamma + a(h-b')]x_u + [d + \gamma_u - \gamma(h-b')]x + (h-b')\bar{x}, \\
\bar{x}_v &= [m' - b'_v + \gamma + b'(h'-a)]x_v + [d' + \gamma_v - \gamma(h'-a)]x + (h'-a)\bar{x}.
\end{aligned}
$$

It follows therefore that *if* $A\neq 0$, $a'=0$, *the surfaces* S *and* $\bar{S}$ *sustain* C *nets, and the lines* g *joining corresponding points* x *and* $\bar{x}$ *form a congruence* G, *the developables of* G *intersecting these surfaces in their* C *nets.*

B. Suppose that $A=0$. Under this condition another integrability condition of system (16) is $b''=0$. Equations (19) may now be written in the form

$$
(22)\quad \begin{aligned}
\bar{x}_u &= (a+\alpha)x_{uu} + [l + \alpha_u + \gamma - \alpha(h-b')]x_u + [d + \gamma_u - \gamma(h-b')]x \\
&\quad + (h-b')\bar{x}, \\
\bar{x}_v &= a'x_{uu} + [l' + \alpha_v - \alpha(h'+\alpha)]x_u + [m' - b'_v + \gamma + b'(h'+\alpha)]x_v \\
&\quad + [d' + \gamma_v - \gamma(h'+\alpha)]x + (h'+\alpha)\bar{x}.
\end{aligned}
$$

It follows that *the tangent to* $v=$const. *on* $\bar{S}$ *intersects the osculating plane to* $v=$const. *on* S. *The tangent planes to* S *and* $\bar{S}$ *at* x *and* $\bar{x}$ *respectively intersect in a point*; *they will intersect in a line if, and only if,* $a'=a+\alpha=0$, *that is, if, and only if, the parametric nets on* S *and* $\bar{S}$ *are in relation* C. In this latter case the lines joining corresponding points x and $\bar{x}$ form a congruence.

3. The parabolic case

Let us consider the parabolic case. If we differentiate $\bar{x}$ defined by (8) with respect to u and v, we obtain

$$
(23)\quad \begin{aligned}
\bar{x}_u &= x_{uuu} + \alpha x_{uu} + \beta x_{uv} + (\alpha_u + \gamma)x_u + \beta_u x_v + \gamma_u x, \\
\bar{x}_v &= x_{uuv} + \alpha x_{uv} + \beta x_{vv} + \alpha_v x_u + (\beta_v + \gamma)x_v + \gamma_v x.
\end{aligned}
$$

It follows therefore that if the points $\bar{x}_u$, $\bar{x}_v$ lie in $S(2, 0)$ the functions x must satisfy a system of differential equations of the form

$$
(24)\qquad \begin{aligned} x_{uuu} &= ax_{uu} + hx_{uv} + bx_{vv} + lx_u + mx_v + dx, \\ x_{uuv} &= a'x_{uu} + h'x_{uv} + b'x_{vv} + l'x_u + m'x_v + d'x. \end{aligned}
$$

It follows therefore *that* $S(3, 0)$ *is of dimensions no higher than seven.*

Subcase a. Suppose that $S(3, 0)$ is of seven dimensions, that is, that the functions x do not satisfy a third third-order differential equation.

The system (24) has the following integrability conditions*:

$$
(25)\qquad \begin{gathered} b = 0,\ h = b',\ a_v = a_u' + a'h' + l', \\ h_v + ah' + l = h_u' + a'h + h'^2 + m', \\ ab' + m = b_u' + b'h',\ l_v + al' = l_u' + a'l + h'l' + d', \\ m_v + am' + d = m_u' + a'm + h'm', \\ d_v + ad' = d_u' + a'd + d'h'. \end{gathered}
$$

It follows from (23) and (24) that the functions $\bar{x}_u$ and $\bar{x}_v$ are defined by the expressions

$$
(26)\qquad \begin{aligned} \bar{x}_u &= (a + \alpha)x_{uu} + (h + \beta)x_{uv} + (l + \alpha_u + \gamma)x_u + (m + \beta_u)x_v + (d + \gamma_u)x, \\ \bar{x}_v &= a'x_{uu} + (h' + \alpha)x_{uv} + (b' + \beta)x_{vv} + (l' + \alpha_v)x_u + (m' + \beta_v + \gamma)x_v \\ &\quad + (d' + \gamma_v)x. \end{aligned}
$$

From (26) we find that the points $\bar{x}_{uu}$, $\bar{x}_{uv}$, $\bar{x}_{vv}$ lie in the space $S(2, 0)$ if and only if

$$
(27)\qquad \alpha + h' = 0, \quad \beta + b' = 0.
$$

Therefore *the surface* $\bar{S}$ *generated by the point* $\bar{x}$ *defined by the expression*

$$
(28)\qquad \bar{x} = x_{uu} - h'x_u - b'x_v + \gamma x
$$

is such that the two-osculating space $\bar{S}(2, 0)$ *at* $\bar{x}$ *coincides with the space* $S(2, 0)$ *at* x *for every choice of* γ.

If we make use of equation (28) we may write equation (26) in the form

$$
(29)\qquad \begin{aligned} \bar{x}_u &= \mu x_u + fx + A\bar{x}, \\ \bar{x}_v &= rx_u + \mu x_v + gx + B\bar{x}, \end{aligned}
$$

wherein

$$
(30)\qquad \begin{gathered} \mu = h'(a - h') + l - h_u' + \gamma = a'b' + m' - b_v' + \gamma, \\ f = d + \gamma_u - \gamma(a - h'), \quad A = a - h', \quad B = a', \\ g = d' + \gamma_v - a'\gamma, \quad r = a'h' + l' - h_v'. \end{gathered}
$$

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* Lane, *Surfaces*, p. 792.

We may readily verify that as x ($\bar{x}$) moves along the curve v = const. on S ($\bar{S}$) the point

$$y = \bar{x} - \mu x, \quad r \neq 0,$$

describes a curve whose tangent at y is the line g joining x to $\bar{x}$. Moreover there exists no other curve on S ($\bar{S}$) along which x ($\bar{x}$) may move so that the line g will generate a developable surface. We may readily verify that *the lines g generate a congruence G composed of the tangents to a one-parameter family of asymptotic curves on the surface generated by the point y.* However the point y defined by the expression

$$y = \bar{x} - \mu x, \quad r = 0,$$

is a fixed point, and the lines g form a bundle of lines through this fixed point.

Subcase b. Suppose that the space $S(3, 0)$ is of six dimensions.

A. The points x_{uv}, x_{vv}, as may be seen from (23), will lie in the space $S(2, 0)$ if

$$\beta = -h, \quad \alpha = -h', \tag{31}$$

and if x satisfies the equations (24) and a differential equation of the form

$$x_{vvv} = a''x_{uu} + h''x_{uv} + b''x_{vv} + l''x_u + m''x_v + d''x. \tag{32}$$

Some of the integrability conditions of the system composed of equations (24) and (32) are

$$b = 0, \; h = b', \; m + b'(a - h') - b_u' = 0.$$

We may readily verify that *the point $\bar{x}$ defined by*

$$\bar{x} = x_{uu} - h'x_u - b'x_v + \gamma x \tag{33}$$

generates a surface $\bar{S}$ whose two-osculating space $\bar{S}(2, 0)$ at $\bar{x}$ coincides with the two-osculating space $S(2, 0)$ of S at x. Moreover *the tangent planes to S at x and $\bar{S}$ at $\bar{x}$ intersect in a line h. The projectivity determined on h by the pencils of tangent lines to S and $\bar{S}$ at x and $\bar{x}$ is parabolic. The lines g joining x to $\bar{x}$ form a congruence of tangents to a one-parameter family of asymptotic curves on a surface.*

B. The space $\bar{S}(2, 0)$ of $\bar{S}$ at $\bar{x}$ will also coincide with the space $S(2, 0)$ at x if

$$\beta + b' = 0,$$

and if x satisfies equations (24) and a differential equation of the form

$$x_{uuv} = a''x_{uu} + h''x_{uv} + b''x_{vv} + l''x_u + m''x_v + d''x. \tag{34}$$

Two of the integrability conditions of such a system are

$$b = 0, \quad b' = 0.$$

It follows therefore that any point defined by the expression

$$\bar{x} = x_{uu} + \alpha x_u + \gamma x$$

(α and γ arbitrary) in the osculating plane to $v=$const. on S at x generates a surface $\bar{S}$ whose two-osculating space $\bar{S}(2,0)$ at $\bar{x}$ coincides with the space $S(2,0)$ at x. The tangent planes to S and $\bar{S}$ at x and $\bar{x}$ intersect in a point.

Suppose that in the expression (4) $A=B=C=0$. By a transformation of the curvilinear coordinates we may write (4) in the form

$$\bar{x} = x_u + \gamma x. \tag{35}$$

By repeated differentiations we find that $\bar{S}(2, 0)$ coincides with $S(2, 0)$ if, and only if, the functions x satisfy a system of differential equations composed of equations of the form (24) and (34). It follows that the space $S(3, 0)$ of S at x is of six dimensions. Conversely if the functions satisfy such a system, a point $\bar{x}$ defined by (35) generates a surface of the required type.

4. The conjugate case

Suppose now that S sustains a conjugate net. By proper choice of the parameters we may take this net to be the parametric net. The functions x therefore satisfy an equation of the Laplace type

$$x_{uv} = ax_u + bx_v + cx. \tag{36}$$

It follows from (36) that $S(2,0)$ is a space of four dimensions and that $S(3,0)$ is a space of not more than six dimensions.

Let the point $\bar{x}$ be defined by the expression

$$\bar{x} = Ax_{uu} + Cx_{vv} + \alpha x_u + \beta x_v + \gamma x, \tag{37}$$

wherein not both A and C are zero.

A. Suppose first that $\bar{S}(3, 0)$ is of six dimensions. We find readily that there exist no surfaces $\bar{S}$ distinct from S such that the spaces $\bar{S}(2, 0)$ and $S(2, 0)$ coincide.

B. Suppose that $S(3, 0)$ is of five dimensions. We find from (37) that

$$\begin{aligned} \bar{x}_u = {} & Ax_{uuu} + (A + \alpha)x_{uu} + (bC + C_u)x_{vv} + [C(a_v + a^2) + a\beta + \alpha_u + \gamma]x_u \\ & + [C(c + ab + b_v) + b\beta + \beta_u]x_v + [C(c_v + ac) + \beta c + \gamma_u]x. \end{aligned} \tag{38}$$

A symmetrical expression obtains for $\bar{x}_v$. It follows that if $A \neq 0$, the functions x satisfy an equation of the form

$$(39)\qquad x_{uuu} = a'x_{uu} + b'x_{vv} + l'x_u + m'x_v + d'x.$$

In order that $\bar{x}_v$ lie in the space $S(2, 0)$, and that $S(3, 0)$ be a space of five dimensions the coëfficient C must be zero.

Some of the integrability conditions of the system composed of equations (36) and (39) are

$$b' = 0, \quad m' = 0, \quad a'_v - a_u = c + ab + a_u.$$

Hence the curves $v =$ const. on S are plane curves. With the expression for $\bar{x}_v$ and $C = 0$, we find that $\bar{x}_{vv}$ lies in $S(2, 0)$ if, and only if, $\beta = 0$. Hence $\bar{x}$ lies in the plane of the curve $v =$ const.

If we set $A = 1$, we find that the points $\bar{x}_u$, $\bar{x}_v$ are defined by the expressions

$$(40)\qquad \begin{aligned} \bar{x}_u = {} & [l' + \gamma + \alpha_u - \alpha(a' + \alpha)]x_u \\ & + [d' + \gamma_u - \gamma(a' + \alpha)]x + (a' + \alpha)\bar{x}, \\ \bar{x}_v = {} & (c + ab + a_u + \alpha_v)x_u + (b^2 + b_u + \alpha b + \gamma)x_v \\ & + (c_u + bc + \alpha c + \gamma_v - a\gamma)x + a\bar{x}. \end{aligned}$$

The tangent planes to S and $\bar{S}$ at x and $\bar{x}$ intersect in a line. Hence *if* $S(3, 0)$ *is a space of five dimensions, and if S sustains a conjugate net, the point $\bar{x}$ defined by* (37) *will describe a surface $\bar{S}$ whose two-osculating space* $S(2, 0)$ *at $\bar{x}$ coincides with* $S(2, 0)$ *at x if and only if each curve of one of the component families of curves of the conjugate net is a plane curve, and the point $\bar{x}$ is a point in the plane of the curve. The lines g joining x and $\bar{x}$ form a congruence.*

Suppose that $\bar{x}$ lies in the tangent plane of S at x, that is, suppose that in (37) $A = C = 0$. We readily verify that if $S(3, 0)$ is of six dimensions the space $\bar{S}(2, 0)$ at $\bar{x}$ cannot coincide with the space $S(2, 0)$ at x for distinct surfaces S and $\bar{S}$. If $S(3, 0)$ is a space of five dimensions, the point $\bar{x}$ must lie in the tangent to one of the curves of the conjugate net, and that family of curves is a family of plane curves.

5. The asymptotic case

Suppose that S sustains a one-parameter family of asymptotic curves. Let the notation be so chosen that the curves $v =$ const. are the asymptotics. It follows that the functions x defining S satisfy the differential equation

$$(41)\qquad x_{uu} = ax_u + bx_v + cx.$$

It follows that the space $S(3, 0)$ is a space of six dimensions at most.

Let $\bar{x}$ be defined by an expression of the form

$$\bar{x} = Bx_{uv} + Cx_{vv} + \alpha x_u + \beta x_v + \gamma x, \tag{42}$$

wherein not both B and C are zero.

A. We may readily verify that if $S(3, 0)$ is a space of six dimensions, there exists no surface $\overline{S}$ distinct from S with the desired property.

B. Suppose therefore that $S(3, 0)$ is a space of five dimensions. It follows from (42) that the points $\bar{x}_u$ and $\bar{x}_v$ are in $S(2, 0)$ if, and only if, $C=0$, and the functions x satisfy a differential equation of the form

$$x_{uvv} = h'x_{uv} + b'x_{vv} + l'x_u + m'x_v + d'x. \tag{43}$$

Two of the integrability conditions of the system composed of equations (41) and (43) are

$$b = 0, \quad c - b_u' + b'(a - b') = 0. \tag{44}$$

It follows therefore that the surface S is ruled.

If in (42) we set $C=0$, $B=1$, we find that

$$\begin{aligned} \bar{x}_u &= (a + \beta)x_{uv} + (a_v + a\alpha + \alpha_u + \gamma)x_u \\ &\quad + (c + \beta_u)x_v + (c_v + \alpha c + \gamma_u)x, \\ \bar{x}_v &= (a' + \alpha)x_{uv} + (\beta + b')x_{vv} + (l' + \alpha_v)x_u \\ &\quad + (m' + \beta_v + \gamma)x_v + (d' + \gamma_v)x. \end{aligned} \tag{45}$$

The points $\bar{x}_{uu}$, $\bar{x}_{uv}$, $\bar{x}_{vv}$ lie in $S(2, 0)$ if, and only if, $\beta = -b'$. Equation (45) may be written in the form

$$\begin{aligned} \bar{x}_u &= [a_v + a\alpha + \alpha_u + \gamma - \alpha(a - b')]x_u + [c_v + \alpha c + \gamma_u - \gamma(a - b')]x \\ &\quad + (a - b')\bar{x}, \\ \bar{x}_v &= [l' + \alpha_v - \alpha(a' + \alpha)]x_u + [m' - b_v' + \gamma + b'(a' + \alpha)]x_v \\ &\quad + [d' + \gamma_v - \gamma(a' + \alpha)]x + (a' + \alpha)\bar{x}. \end{aligned} \tag{46}$$

The point $\bar{x}$ defined by the expression

$$\bar{x} = x_{uv} + \alpha x_u - b'x_v + \gamma x$$

for arbitrary values of α and γ generates a surface $\overline{S}$ whose two-osculating space $\overline{S}(2, 0)$ at $\bar{x}$ coincides with the two-osculating space $S(2, 0)$ of S at x.

The point r defined by the expression $r = x_u - b'x$ is readily characterized as the only point, on the generator through x of the ruled surface, describing a surface for which the osculating plane to the curve $u=$const. at r lies in the space of three dimensions tangent to the ruled surface along the generator through x. We find that

$$r_v + \alpha r = \bar{x} - (\alpha b' + \gamma + b_v')x.$$

It follows that *the lines g joining x to $\bar{x}$ form a congruence. The line g passes through x and intersects the tangent line to the curve u* = const. *on the surface generated by the point r.*

Suppose that $\bar{x}$ lies in the tangent plane to S at x. We readily verify that $\bar{S}(2, 0)$ at $\bar{x}$ will coincide with $S(2, 0)$ at x if and only if $\bar{x}$ lies in the tangent line of the asymptotic curve on S through x, and if the functions x defining the surface satisfy a differential equation of the form (43).

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